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# Toward a $q$-deformed standard model ★

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**Abstract**

A gauge theory with an underlying $SU_q(2)$ quantum group symmetry is introduced, and its properties are examined. With suitable assumptions, this model is found to have many similarities with the usual $SU(2) \times U(1)$ standard model, specifically, the existence of four generators and thus four gauge fields. However, the two classical symmetries are unified into one quantum symmetry, and therefore there is only a single coupling constant, rather than two. By incorporating a Higgs sector into the model, one obtains several explicit tree-level predictions in the undeformed limit, such as the Weinberg angle: $\sin^2\theta_W = \frac{3}{11}$. With the $Z$-boson mass $m_Z$ and fine structure constant $\alpha$ as inputs, one can also obtain predictions for the weak coupling constant, the mass of the $W$, and the Higgs VEV. The breaking of the quantum invariance also results in a remaining undeformed $U(1)$ gauge symmetry.

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## 1. Introduction

It has long been known that symmetries play a vital role in nature, and that any physical theory describing observations must take these into account. This attitude has yielded amazing success, perhaps the most notable example being the standard model (SM), in which it is assumed that reality has a $SU(2) \times U(1)$ symmetry.

This symmetry group is a so-called "classical" group, i.e. it satisfies the usual properties of closure, existence of inverses and an identity, and associativity. There is, however, a more

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general notion which includes the classical case, that of a matrix pseudogroup, or more familiarly, a quantum group (QG) [1,2]. This object is a "deformed" version of the classical case, in much the same way quantum mechanics is a "deformed" version of Newtonian mechanics (hence the moniker "quantum group"). As such, one usually thinks of the deviation of a QG from the usual case as parametrized by a number $q$, with $q = 1$ giving the classical case, just as $\hbar$ encapsulates the difference between classical and quantum mechanics, with the correspondence principle allowing one to relate the two. Therefore, since the deformation of classical mechanics into quantum mechanics results in new physics, it would not be surprising to expect the same to occur in a theory with a QG symmetry.

In the SM, the symmetry group is taken to be local, and therefore must be gauged, with the four particles needed to do this being identified with the photon $A$ and the three weak-force mediators $W^{\pm}$ and $Z$. However, the overall symmetry is also assumed to be broken spontaneously via a Higgs mechanism, leading to a unbroken $U(1)$ symmetry group and masses for the weak bosons.

Since this approach is so remarkably successful in describing nature, if one is interested in deforming the SM, it seems reasonable to incorporate the same concepts into a theory with a QG symmetry. This paper attempts to do just that: To construct a gauge theory with a symmetry group corresponding to the deformed version of $SU(2)$, denoted by $SU_q(2)$. Not surprisingly, other authors have also pursued this line of thought (with [3,4] being particularly relevant to this work). However, this work approaches the problem via the deformation of the classical Lie algebra $\mathfrak{su}(2)$ into the quantum Lie algebra (QLA) [5,6] $U_q\mathfrak{su}(2)$), and then uses the fact that $U_q(\mathfrak{su}(2))$ has *four* generators, and therefore needs the same number of gauge fields as the classical symmetry algebra $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ does. Furthermore, there will be only *one* coupling constant needed, since there is only one symmetry.

The approach presented here treads a thin line, perhaps. It is necessary to be somewhat abstract in order to discuss connections algebraically. In fact, this subject should technically be treated within the context of sheaf theory: In the classical case (cf. [7]), this allows topological properties of fiber bundles to be translated into algebraic language, and there has been substantial progress in generalizing the same basic theory to the deformed case as well [8]. (There have also been some nice treatments of deformed gauge theories without resorting to sheaves, e.g. [9].) However, since this work means to deal with more physical aspects, the ultimate goal of interpreting these objects as gauge fields is kept in mind, so some of the more mathematical details will be deemphasized or ignored.

Finally, the author must stress that the model presented here is *highly* speculative. As the reader will see, there are plenty of assumptions and leaps of faith made, some perhaps unjustifiably, in order to obtain heuristic results. In fact, it is not clear that this particular approach is even the best, since much work has already been done in the subject by emphasizing the noncommutative geometric nature of the SM rather than the QG aspect [10,11]. However, if one accepts the formal existence of the objects appearing through this work, then everything is mathematically consistent. Furthermore, all the predictions are given only for the undeformed case anyway, where there are explicit and well-understood expressions for everything (for example, the quadratic form introduced in Section 4.2 simply becomes the usual integral at $q = 1$). In any case, this model has some interesting features, and the author's purpose in presenting the material herein is to introduce what may perhaps be a starting point for further explorations.

A note: Although later sections will refer to some of the notations and definitions in Sections 2 and 3, it is not absolutely necessary for the reader to spend an excessive amount of time on these sections, and s/he can move on to Section 4.3 and, if still interested, come back later for the mathematical details.

## 2. Deformed gauge theories

In order to generalize the concepts involved in discussing a deformed gauge theory, one must use the language of Hopf algebras (HAs) [12–14], as is usually the case when talking about QGs. As a result, this section will be rather abstract and mathematical.

Let $\mathcal{M}$ be a unital associative *-algebra and $\mathcal{A}$ a *-Hopf algebra (both over a field $k$) which (left) coacts on $\mathcal{M}$ [6], i.e. there exists a linear algebra map ${}_{\mathcal{A}}\Delta : \mathcal{M} \to \mathcal{A} \otimes \mathcal{M}$, $\phi \mapsto \phi^{(1)'} \otimes \phi^{(2)}$, satisfying the following:

$$\begin{aligned} &\phi^{(1)'} \otimes {}_{\mathcal{A}}\Delta(\phi^{(2)}) = \Delta(\phi^{(1)'}) \otimes \phi^{(2)}, \qquad \epsilon(\phi^{(1)'})\phi^{(2)} = \phi, \\ &{}_{\mathcal{A}}\Delta(\overline{\phi}) = \theta(\phi^{(1)'}) \otimes \overline{\phi^{(2)}}, \qquad {}_{\mathcal{A}}\Delta(1) = 1 \otimes 1, \end{aligned} \tag{2.1}$$

where $\Delta$, $\epsilon$ and $\theta$ are, respectively, the coproduct, counit, and involution on $\mathcal{A}$, and $^{-}$ the involution on $\mathcal{M}$. (A right coaction $\Delta_{\mathcal{A}} : \mathcal{M} \to \mathcal{M} \otimes \mathcal{A}$ may be defined analogously.)

Now, suppose that $\delta$ is an exterior derivative on $\mathcal{A}$, and from this we construct the universal differential calculus $(\Omega(\mathcal{A}), \delta)$, with $\Omega(\mathcal{A})$ being the universal differential envelope (UDE) of $\mathcal{A}$ [15,16]. This is given a $\mathbb{Z}_2$-graded HA structure in accordance with [17]. We similarly assume that d is an exterior derivative on $\mathcal{M}$, and take $\Omega(\mathcal{M})$ as the UDE (with the multiplication in this algebra denoted by $\wedge$). The coaction above extends to a map on these UDEs, i.e. ${}_{\mathcal{A}}\Delta : \Omega^p(\mathcal{M}) \to \bigoplus_{q=0}^{p} \Omega^{p-q}(\mathcal{A}) \otimes \Omega^q(\mathcal{M})$, defined recursively by

$${}_{\mathcal{A}}\Delta(\mathrm{d}\phi) \equiv \delta\phi^{(1)'} \otimes \phi^{(2)} + (-1)^{|\phi^{(1)'}|}\phi^{(1)'} \otimes \mathrm{d}\phi^{(2)}, \tag{2.2}$$

where $|\phi|$ is the degree of $\phi$, e.g. $p$ if $\phi \in \Omega^p(\mathcal{M})$.

### 2.1. Connections, field strengths and covariant derivatives

A connection on $\mathcal{M}$ is defined as a linear map $\Gamma : \Omega^p(\mathcal{A}) \to \Omega^{p+1}(\mathcal{M})$ satisfying the following:

$$\begin{aligned} \Gamma(1) &= 0, \\ \Gamma(\delta\alpha) &= -\mathrm{d}\Gamma(\alpha), \\ {}_{\mathcal{A}}\Delta(\Gamma(\alpha)) &= (-1)^{|\alpha_{(1)}|+|\alpha_{(3)}|(|\alpha_{(2)}|+1)}\alpha_{(1)}S(\alpha_{(3)}) \otimes \Gamma(\alpha_{(2)}) \\ &\quad - \delta\alpha_{(1)}S(\alpha_{(2)}) \otimes 1 \end{aligned} \tag{2.3}$$

($\alpha \in \Omega(\mathcal{A})$). This in turn allows the definition of the field strength $F : \Omega^p(\mathcal{A}) \to \Omega^{p+2}(\mathcal{M})$ via

$$F(\alpha) := \mathrm{d}\Gamma(\alpha) + (-1)^{|\alpha_{(1)}|}\Gamma(\alpha_{(1)}) \wedge \Gamma(\alpha_{(2)}). \tag{2.4}$$

The last of (2.3) then implies that

$$_{\mathcal{A}}\Delta(F(\alpha)) = (-1)^{|\alpha_{(2)}||\alpha_{(3)}|}\alpha_{(1)}S(\alpha_{(3)}) \otimes F(\alpha_{(2)}). \tag{2.5}$$

Let $\psi \in \Omega^p(\mathcal{M})$ be a $p$-form on $\mathcal{M}$; one can define a covariant derivative D as

$$\mathrm{D}\psi := \mathrm{d}\psi + \Gamma(\psi^{(1)'}) \wedge \psi^{(2)} \tag{2.6}$$

so that D maps $p$-forms into $(p+1)$-forms. It follows immediately from the coaction of $\mathcal{A}$ on $\Gamma$ that

$$_{\mathcal{A}}\Delta(\mathrm{D}\psi) = (-1)^{|\psi^{(1)'}|}\psi^{(1)'} \otimes \mathrm{D}\psi^{(2)}, \tag{2.7}$$

which is exactly what one would want for a covariant derivative. This, in turn, gives $\mathrm{D}^2\psi \equiv F(\psi^{(1)'}) \wedge \psi^{(2)}$, as in the classical case.

To illustrate how this covariant derivative works, consider the following example: Suppose that $\mathcal{A}$ is a QG, and $\psi^i$ a form living in the fundamental rep of $\mathcal{A}$ so that if $A^i{}_j$ is the QG matrix associated with this rep,

$$_{\mathcal{A}}\Delta(\psi^i) := A^i{}_j \otimes \psi^j; \tag{2.8}$$

then if $\Gamma^i{}_j := \Gamma(A^i{}_j)$,

$$_{\mathcal{A}}\Delta(\Gamma^i{}_j) = A^i{}_k S(A^\ell{}_j) \otimes \Gamma^k{}_\ell - \delta A^i{}_k S(A^k{}_j) \otimes 1, \tag{2.9}$$

and

$$\mathrm{D}\psi^i \equiv \mathrm{d}\psi^i + \Gamma^i{}_j \wedge \psi^j \mapsto A^i{}_j \otimes \mathrm{D}\psi^j. \tag{2.10}$$

Notice that the field strength $F^i{}_j := \mathrm{d}\Gamma^i{}_j + \Gamma^i{}_k \wedge \Gamma^k{}_j$ is thus a 2-form on $\mathcal{M}$ transforming according to

$$_{\mathcal{A}}\Delta(F^i{}_j) = A^i{}_k S(A^\ell{}_j) \otimes F^k{}_\ell. \tag{2.11}$$

Let $\mathcal{U}$ be the HA dually paired with $\mathcal{A}$; if $\mathcal{A}$ is a QG generated by the matrix entries $\{A^i{}_j | i, j = 1, \ldots, n\}$, then $\mathcal{U}$ is a QLA generated by

$$X^i{}_j \equiv T_{(ij)} := \frac{1}{\lambda}\langle 1 \otimes 1 - \mathcal{R}_{21}\mathcal{R}, A^i{}_j \otimes \mathrm{id}\rangle, \tag{2.12}$$

where $\langle\ ,\ \rangle$ is the dual pairing between $\mathcal{U}$ and $\mathcal{A}$, $\mathcal{R} \in \mathcal{U} \otimes \mathcal{U}$ the universal R-matrix, and $\lambda$ is the frequently occurring expression $q - q^{-1}$. One can require that the connection, when the argument is in $\Omega^0(\mathcal{A}) \equiv \mathcal{A}$, takes the form

$$\Gamma(a) \equiv \Gamma^A \langle T_A, a\rangle = \Gamma^{(ij)}\langle T_{(ij)}, a\rangle, \tag{2.13}$$

where $\Gamma^A \in \Omega^1(\mathcal{M})$ for $A = 1, \ldots, n^2$. The motivation for this particular definition comes from the following two facts: First, the exterior derivative $\delta$ on $\mathcal{A}$ may be used to define a basis $\{\omega^A\}$ for $\Omega^1(\mathcal{A})$ via

$$\delta a \equiv \omega^A \langle T_A, a_{(1)} \rangle a_{(2)}; \tag{2.14}$$

secondly, with the quantum matrices $\mathbb{A}^A{}_B$

$$\mathbb{A}^{(ij)}{}_{(k\ell)} = S(A^k{}_i) A^j{}_\ell \tag{2.15}$$

giving the adjoint representation of the QLA via the right coaction of the generators [18], i.e.

$$\Delta_{\mathcal{A}}(T_A) = T_B \otimes \mathbb{A}^B{}_A, \tag{2.16}$$

then it may be shown that by using ${}_{\mathcal{A}}\Delta(\Gamma(a)) = {}_{\mathcal{A}}\Delta(\Gamma^A)\langle T_A, a \rangle$, one obtains

$${}_{\mathcal{A}}\Delta(\Gamma^A) \equiv \mathbb{A}^A{}_B \otimes \Gamma^B - \omega^A \otimes 1, \tag{2.17}$$

the usual inhomogeneous transformation law for the components of $\Gamma$.

The above coaction will of course give the transformation properties of the field strength; notice, however, that the definition of $F$ implies

$$F(a) = \mathrm{d}\Gamma^A \langle T_A, a \rangle + \Gamma^A \wedge \Gamma^B \langle T_A T_B, a \rangle. \tag{2.18}$$

Unlike the classical case, this does *not* necessarily have the form $F^A \langle T_A, a \rangle$; when the QLA is undeformed, the connection 1-forms anticommute, and $T_A T_B$ can be replaced by $\frac{1}{2}[T_A, T_B]$. This is equivalent to $\frac{1}{2} f_{AB}{}^C T_C$, and $F$ would be $\mathfrak{g}$-valued. Here, in the deformed situation, one must *impose* appropriate anticommutation relations on the connections in order for this to be the case. If this is done, then the field strength is found to transform homogeneously according to

$${}_{\mathcal{A}}\Delta(F^A) = \mathbb{A}^A{}_B \otimes F^B, \tag{2.19}$$

just as in the classical case.

## 3. $SU_q(2)$

Since the model presented here is considered to have a deformed $SU_q(2)$ symmetry, some details about this QG (and its associated QLA $U_q(\mathfrak{su}(2))$) must first be mentioned.

First of all, for any quasitriangular HA with a given universal R-matrix, it is possible to construct the corresponding QLA [19,20]. $U_q(\mathfrak{su}(2))$ is such a case [21]; however, for calculational purposes, the more important object is the $4 \times 4$ numerical R-matrix for the quantum group $SU_q(2)$; this is easily obtained from the general form for the $SL_q(N)$ R-matrix [22,23] with $N = 2$, and with unitarity implying $q \in \mathbb{R}$:

$$R = q^{-1/2}\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}. \tag{3.1}$$

This is simply the universal R-matrix in the fundamental representation, i.e. $R^{ij}{}_{k\ell} := \langle \mathcal{R}, U^i{}_k \otimes U^j{}_\ell \rangle$, $U$ being the $2 \times 2$ $SU_q(2)$ quantum matrix satisfying $RU_1U_2 = U_2U_1R$. The $2 \times 2$ matrix $X$ is given by (2.12) with $A = U$, with the generators $T_1$, $T_+$, $T_-$ and $T_2$ defined as

$$X := \frac{1}{\lambda}\langle 1 \otimes 1 - \mathcal{R}_{21}\mathcal{R}, U \otimes \mathrm{id} \rangle = \begin{pmatrix} T_1 & T_+ \\ T_- & T_2 \end{pmatrix}. \tag{3.2}$$

The hermiticity condition which must be imposed on the generators of $U_q(\mathfrak{su}(N))$ is $\theta(T_{(ij)}) = T_{(ji)}$, and for $N = 2$, implies that $T_1$ and $T_2$ are self-adjoint, and $\theta(T_\pm) = T_\mp$.

For the discussion presented here, a more convenient basis turns out to be $T_0 = T_1 + (1/q^2)T_2$, $T_3 := [q^2/(1+q^2)](T_1 - T_2)$, and $T_\pm$. Changing to these particular linear combinations allows one to see more transparently the relation between the deformed and undeformed cases. For example, the QLA versions of the adjoint action $\overset{\mathrm{ad}}{\triangleright}$ of $U_q(\mathfrak{su}(2))$ on itself, written in terms of the structure constants via $T_A \overset{\mathrm{ad}}{\triangleright} T_B = f_{AB}{}^C T_C$ [6], are

$$T_0 \overset{\mathrm{ad}}{\triangleright} T_0 = 0,\ T_a \overset{\mathrm{ad}}{\triangleright} T_0 = 0,\ T_0 \overset{\mathrm{ad}}{\triangleright} T_a = -\lambda\,[2]\,T_a \tag{3.3}$$

(where $a = \pm, 3$), as well as

$$\begin{aligned} T_3 \overset{\mathrm{ad}}{\triangleright} T_3 &= -\lambda T_3, & T_\pm \overset{\mathrm{ad}}{\triangleright} T_\mp &= \pm\frac{[2]}{q}T_3, \\ T_3 \overset{\mathrm{ad}}{\triangleright} T_\pm &= \pm q^{\mp 1} T_\pm, & T_\pm \overset{\mathrm{ad}}{\triangleright} T_3 &= \mp q^{\pm 1} T_\pm, \end{aligned} \tag{3.4}$$

where the "quantum number" $[n]$ is defined as

$$[n] := \frac{1 - q^{-2n}}{1 - q^{-2}}. \tag{3.5}$$

These of course reduce to the usual $\mathfrak{su}(2)$ commutators in the $q \to 1$ limit.

The commutation relations themselves may also be found, and one finds that $T_0$ is central, and the other generators satisfy

$$\begin{aligned} q^{\mp 1} T_3 T_\pm - q^{\pm 1} T_\pm T_3 &= \pm\left(1 - \frac{\lambda}{[2]}T_0\right)T_\pm, \\ T_+T_- - T_-T_+ &= \frac{[2]}{q}\left(1 - \frac{\lambda}{[2]}T_0\right)T_3 + \frac{\lambda\,[2]}{q}T_3^2. \end{aligned} \tag{3.6}$$

There is also the identity

$$\left(1 - \frac{\lambda}{[2]}T_0\right)^2 = 1 + q^2\lambda^2 J^2, \tag{3.7}$$

where

$$J^2 := \frac{1}{q^2\,[2]}(q^2 T_+ T_- + T_- T_+ + [2]\,T_3^2) \tag{3.8}$$

is the $U_q(\mathfrak{su}(2))$ quadratic casimir. Again, the classical limits of the above give the familiar commutation relations.

The general definition of the $n \times n$ fundamental rep for a QLA is

$$\underline{\mathrm{fn}}^i{}_j(T_{(k\ell)}) := \langle T_{(k\ell)}, U^i{}_j \rangle = \frac{1}{\lambda}(I - R_{21}R)^{ki}{}_{\ell j}, \tag{3.9}$$

so, by using (3.1), the numerical matrices for the generators $\{T_1, T_+, T_-, T_2\}$ in the $2 \times 2$ fundamental rep $\underline{\mathrm{fn}}$ of $U_q(\mathfrak{su}(2))$ may be found, and when the change of basis above is said and done, one gets

$$\begin{aligned} &\underline{\mathrm{fn}}(T_0) = -\frac{\lambda}{q}\left[\frac{1}{2}\right]\left[\frac{3}{2}\right]\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \underline{\mathrm{fn}}(T_3) = \frac{1}{[2]}\begin{pmatrix} -1 & 0 \\ 0 & 1/q^2 \end{pmatrix}, \\ &\underline{\mathrm{fn}}(T_+) = \begin{pmatrix} 0 & 0 \\ -1/q & 0 \end{pmatrix}, \qquad \underline{\mathrm{fn}}(T_-) = \begin{pmatrix} 0 & -1/q \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{3.10}$$

(Note that $T_0$ vanishes as $q \to 1$; this is what gets rid of this extra generator for classical $\mathfrak{su}(2)$.)

The adjoint rep $\underline{\mathrm{ad}}$ follows immediately from the adjoint actions given in (3.3) and (3.4), and in the basis $\{T_0, T_+, T_-, T_3\}$, they take the forms

$$\begin{aligned} &\underline{\mathrm{ad}}(T_0) = -\lambda\,[2]\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \underline{\mathrm{ad}}(T_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/q & 0 & 0 \\ 0 & 0 & -q & 0 \\ 0 & 0 & 0 & -\lambda \end{pmatrix}, \\ &\underline{\mathrm{ad}}(T_+) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -q\,[2] \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/q & 0 \end{pmatrix}, \qquad \underline{\mathrm{ad}}(T_-) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & [2]/q \\ 0 & -1/q & 0 & 0 \end{pmatrix}. \end{aligned} \tag{3.11}$$

## 4. A deformed standard model

### 4.1. $U_q(\mathfrak{su}(2))$ connection and field strengths

Recall that in general $F(a)$ does not in general have the form $F^A\langle T_A, a\rangle$. However, $F$ may be written in this way if one is willing to *impose* ad hoc a certain set of anticommutation relations on the connection 1-forms, which can be determined for the case where the QLA being considered is $U_q(\mathfrak{su}(2))$. By using the generator commutation relations (3.6) and the identity (3.7), it is straightforward to see that the deformed anticommutation relations

$$
\begin{aligned}
&\Gamma^{\pm} \wedge \Gamma^{\pm} = 0, \\
&\Gamma^{\pm} \wedge \Gamma^{3} + q^{\pm 2} \Gamma^{3} \wedge \Gamma^{\pm} = 0, \\
&\Gamma^{\pm} \wedge \Gamma^{0} + \Gamma^{0} \wedge \Gamma^{\pm} = \pm \frac{q^{\pm 1}\lambda}{[2]} \Gamma^{3} \wedge \Gamma^{\pm}, \\
&\Gamma^{+} \wedge \Gamma^{-} + \Gamma^{-} \wedge \Gamma^{+} = -q^{2}\,[2]^{2}\, \Gamma^{0} \wedge \Gamma^{0}, \\
&\Gamma^{0} \wedge \Gamma^{3} + \Gamma^{3} \wedge \Gamma^{0} = -\frac{\lambda}{q} \Gamma^{-} \wedge \Gamma^{+} - \frac{\lambda\,[2]}{q} \Gamma^{0} \wedge \Gamma^{0}, \\
&\Gamma^{3} \wedge \Gamma^{3} = \frac{\lambda\,[2]}{q} \Gamma^{-} \wedge \Gamma^{+} - \frac{[2]^{2}}{q^{2}} \Gamma^{0} \wedge \Gamma^{0},
\end{aligned}
\tag{4.1}
$$

do indeed make $F$ $U_q(\mathfrak{su}(2))$-valued, with components

$$
\begin{aligned}
&F^{0} = \mathrm{d}\Gamma^{0} + \frac{2\,[2]}{\lambda} \Gamma^{0} \wedge \Gamma^{0}, \\
&F^{\pm} = \mathrm{d}\Gamma^{\pm} \pm q^{\pm 1} \Gamma^{3} \wedge \Gamma^{\pm}, \\
&F^{3} = \mathrm{d}\Gamma^{3} - \frac{[2]}{q} \Gamma^{-} \wedge \Gamma^{+} - \frac{[2]^{2}}{q} \Gamma^{0} \wedge \Gamma^{0}.
\end{aligned}
\tag{4.2}
$$

All the above relations may be simplified with another ad hoc assumption: $\Gamma^0 \wedge \Gamma^0 \equiv 0$. This is certainly true in the undeformed case, since $\Gamma^0$ is simply a 1-form; however, one must make sure that this assumption is consistent with the deformed anticommutation relations (4.1). This is easily shown; using these relations, one finds $\Gamma^0 \wedge \Gamma^0$ commutes with each 1-form. As for covariance under the coaction ${}_{\mathcal{A}}\Delta$, notice from (2.17) that the transformation for the connection component $\Gamma^0$ is

$$
{}_{\mathcal{A}}\Delta(\Gamma^{0}) = 1 \otimes \Gamma^{0} - \omega^{0} \otimes 1. \tag{4.3}
$$

Since multiplication on the tensor product space $\Omega(\mathcal{A}) \otimes \Omega(\mathcal{M})$ is $\mathbb{Z}_2$-graded, i.e.

$$
(\alpha \otimes \phi)(\beta \otimes \psi) := (-1)^{|\phi||\beta|} \alpha\beta \otimes \phi\psi, \tag{4.4}
$$

it immediately follows that

$$
{}_{\mathcal{A}}\Delta(\Gamma^{0} \wedge \Gamma^{0}) = 1 \otimes (\Gamma^{0} \wedge \Gamma^{0}) + (\omega^{0} \wedge \omega^{0}) \otimes 1. \tag{4.5}
$$

It was proven in [23] that $\omega^0$ is nilpotent, [2] so $\Gamma^0 \wedge \Gamma^0$ is a left-invariant 2-form. Thus, the nilpotency of $\Gamma^0$ will be preserved under the coaction of $\mathcal{A}$, so the anticommutation relations (4.1) remain covariant even with this assumption, and this will not break the $SU_q(2)$ symmetry. (It should again be emphasized that this nilpotency is an assumption, made purely to facilitate what follows.)

The commutation relations between the connections and their exterior derivatives may be defined to be those such that the Bianchi identities hold, i.e. $\mathrm{D}F^A \equiv \mathrm{d}F^A + f_{BC}{}^{A}\Gamma^B \wedge F^C$ vanishes identically. This assumption is yet another one put in by hand, but has the great advantage that the resulting relations are automatically covariant. These may be found in

[2] $\omega^0 = (1/[N])\xi$ for $SU_q(N)$, where $\xi$ is the 1-form of [23].

the Appendix A, and lead to the following commutation relations for the components of the field strength:

$$\begin{aligned} &F^3 \wedge F^{\pm} - q^{\pm 2} F^{\pm} \wedge F^3 = \pm q^{\pm 1} \lambda\, [2]\, F^0 \wedge F^{\pm}, \\ &F^+ \wedge F^- - F^- \wedge F^+ = q\lambda F^0 \wedge F^3 + \frac{q\lambda}{[2]} F^3 \wedge F^3, \\ &F^0 \wedge F^a = F^a \wedge F^0. \end{aligned} \tag{4.6}$$

(The closure of the algebra of the field strength components is a consequence of general covariance; unlike the connection components, the $F$s transform homogeneously in the adjoint rep.)

So far, nothing has been said about the hermiticity of the connection components $\Gamma^0$, $\Gamma^{\pm}$ and $\Gamma^3$. Since in physics one likes to know about adjoints and conjugates, and these connections will ultimately be identified with the $SU_q(2)$ gauge fields, it would be nice to address this subject.

Note the following: At the level of the quantum group $SU_q(2)$, the unitarity condition for the matrices $U^i{}_j$ in the fundamental rep is given by $S(U) = U^{\dagger} = \theta(U^{\mathrm{T}})$, just as in the classical case (recall that $S(U) = U^{-1}$). Therefore, if $\Gamma = \Gamma^A \underline{\mathrm{fn}}(T_A)$ is the 1-form-valued $2 \times 2$ matrix, then note that by using the HA identity $\langle \theta(x), a \rangle \equiv \langle x, (\theta \circ S)(a) \rangle^*$ for $x \in \mathcal{U}$, $a \in \mathcal{A}$, then

$$\Gamma^{\dagger} = \overline{\Gamma^{(ij)}} \underline{\mathrm{fn}}(T_{(ji)}). \tag{4.7}$$

So if $\Gamma$ (and therefore the $2 \times 2$ field strength matrix $F$) is antihermitean, as is usually required, this implies that $\overline{\Gamma^{(ij)}} = -\Gamma^{(ji)}$, so $\Gamma^0$ and $\Gamma^3$ are antihermitean, and $\overline{\Gamma^{\pm}} = -\Gamma^{\mp}$, and, consequently, the same for the field strength components.

### 4.2. $SU_q(2)$ *Yang–Mills action*

In order to use what has been developed so far to build a physical theory, more conditions must be met: First of all, assume the existence of a quadratic form on the differential algebra $\Omega(\mathcal{M})$, i.e. a map $\langle\ |\ \rangle : \Omega(\mathcal{M}) \otimes \Omega(\mathcal{M}) \to k$ (e.g. $\langle \phi | \psi \rangle := \int_M \phi \wedge \star \psi$ for a Riemannian manifold $M$, $\phi$ and $\psi$ $p$-forms on $M$, $k = \mathbb{C}$ for $q = 1$). Furthermore, this form is understood to respect the coaction of $\mathcal{A}$ on $\mathcal{M}$, i.e. under ${}_{\mathcal{A}}\Delta$,

$$\langle \phi | \psi \rangle \mapsto \phi^{(1)'} \psi^{(1)'} \langle \phi^{(2)} | \psi^{(2)} \rangle \tag{4.8}$$

(so the quadratic form is not necessarily symmetric, since in general $\mathcal{A}$ is noncommutative). Consistency with the involutions on $\mathcal{A}$, $\mathcal{M}$ and $k$ also requires

$$\langle \phi | \psi \rangle^* = \langle \bar{\psi} | \bar{\phi} \rangle. \tag{4.9}$$

The Killing metric for an arbitrary QLA was examined in detail in [18], and now enters into the picture. Recall the definition: If $\rho$ is a rep of a QLA with generators $\{T_A | A = 1, \ldots, n^2\}$, then the $n^2 \times n^2$ numerical Killing metric is

$$\eta^{(\rho)}_{AB} := \mathrm{tr}_{\rho}(u T_A T_B) \tag{4.10}$$

$$g := q\kappa\sqrt{\frac{[2]}{[4]}}. \tag{4.16}$$

From what was previously discussed about the antihermiticity of the connections, it follows that $B$ and $W^3$ are self-adjoint and $\overline{W^{\pm}} = W^{\mp}$. Furthermore, the Yang–Mills action now takes the form

$$\begin{aligned}
S_{\mathrm{YM}} = {} & \frac{1}{[2]}\langle \mathrm{d}W^+|\mathrm{d}W^-\rangle + \frac{1}{q^2\,[2]}\langle \mathrm{d}W^-|\mathrm{d}W^+\rangle + \frac{1}{2}\langle \mathrm{d}W^3|\mathrm{d}W^3\rangle \\
& + \frac{\mathrm{i}g}{q[2]}(\langle \mathrm{d}W^+|W^3\wedge W^-\rangle - \langle \mathrm{d}W^-|W^3\wedge W^+\rangle \\
& + \langle \mathrm{d}W^3|W^-\wedge W^+\rangle + \frac{1}{q^2}\langle W^3\wedge W^-|\mathrm{d}W^+\rangle \\
& - q^2\langle W^3\wedge W^+|\mathrm{d}W^-\rangle + \langle W^-\wedge W^+|\mathrm{d}W^3\rangle) \\
& + \frac{g^2}{q\,[2]}(\langle W^3\wedge W^+|W^3\wedge W^-\rangle + \frac{1}{q^2}\langle W^3\wedge W^-|W^3\wedge W^+\rangle \\
& - \frac{2}{q\,[2]^2}\langle W^-\wedge W^+|W^-\wedge W^+\rangle) + \frac{1}{2}\langle \mathrm{d}B|\mathrm{d}B\rangle,
\end{aligned} \tag{4.17}$$

Thus, in the $q \to 1$ limit, $S_{\mathrm{YM}}$ is the usual SM action for the gauge fields $W^{\pm}$, $W^3$ and $B$.

### *4.3. Higgs mechanism, Weinberg angle and gauge field masses*

The Yang–Mills action (4.17) was explicitly constructed to be invariant under the deformed symmetry group $SU_q(2)$, and bears a definite resemblance to the YM term present in the $SU(2)\times U(1)$-symmetric SM. In fact, in the $q \to 1$ limit, the two agree exactly. However, one might argue that this is not a profound result, since $SU_q(2)$ is algebraically equivalent to $SU(2)\times U(1)$ anyway; nothing new is really happening.

However, although the two groups are indeed the same at the *algebraic* level, they are not at the *Hopf algebraic* level, due to the fact that the additional structure (e.g. the coproduct) mixes the "$SU(2)$" part given by $\{T_{\pm}, T_3\}$ and the central "$U(1)$" piece from $T_0$. It is this mixing which changes the situation drastically: In the undeformed SM, one is free to pick the normalizations of the $SU(2)$ and $U(1)$ gauge fields arbitrarily, since the overall symmetry is just the product of the two groups. However, in order to keep the *quantum* symmetry, i.e. the HA structure intact, the relative sizes of the $1\times 1$ and $3\times 3$ pieces of the Killing metric $\eta^{(\rho)}_{00}$ and $\eta^{(\rho)}_{ab}$ are fixed once the rep $\rho$ is picked. The relative normalizations of the $W$s and $B$ are therefore restricted if one requires that this action becomes the familiar YM action in the classical limit. Furthermore, the existence of one, rather than two, symmetry groups explains why only the one coupling constant is present, rather than the two appearing in the undeformed SM.

The consequences of this become manifest when one considers the coupling of the gauge fields to matter. Ordinarily, one picks this matter to live in a specific rep of the symmetry group, and then finds the interactions with the gauge fields via the covariant derivative, and the situation is no different here. All resulting matter-gauge interactions will depend only on $g$.

To illustrate this, consider a complex matter doublet $\Phi^i \in \Omega^0(\mathcal{M})$ (and its conjugate $\Phi_i^\dagger := \overline{\Phi^i}$) living in the fundamental rep of $SU_q(2)$, i.e.

$$\Phi := \begin{pmatrix} \phi^- \\ \phi^0 \end{pmatrix}, \qquad \Phi^\dagger := \begin{pmatrix} \phi^+ & \bar{\phi}^0 \end{pmatrix}. \tag{4.18}$$

Under the QG action, these transform, respectively, as

$$\Phi^i \mapsto U^i{}_j \otimes \Phi^j, \qquad \Phi_i^\dagger \mapsto S(U^j{}_i) \otimes \Phi_j^\dagger. \tag{4.19}$$

Not surprisingly, since the entries of $U$ do not commute, the requirement that the commutation relations between the $\phi$s be covariant under the above coactions implies that they too are deformed, and look like

$$\begin{aligned} \phi^0\phi^\pm &= \frac{1}{q}\phi^\pm\phi^0, & \bar{\phi}^0\phi^\pm &= q\phi^\pm\bar{\phi}^0, \\ \phi^+\phi^- &= \phi^-\phi^+, & \bar{\phi}^0\phi^0 &= \phi^0\bar{\phi}^0 - \frac{\lambda}{q}\phi^+\phi^-. \end{aligned} \tag{4.20}$$

It follows that the quantity $\Phi^\dagger\Phi := \overline{\Phi^i}\Phi^i \equiv \bar{\phi}^0\phi^0 + \phi^+\phi^-$ is central and invariant. Therefore, the appropriate kinetic energy term for this matter will be $\langle (\mathrm{D}\Phi)^\dagger | \mathrm{D}\Phi \rangle$. $\Phi$ lives in the fundamental, so its covariant derivative is given by $\mathrm{D}\Phi := \mathrm{d}\Phi + \Gamma^A \underline{\mathrm{fn}}(T_A)\Phi$; using (3.10) and (4.15),

$$\begin{aligned} \mathrm{D}\phi^- &= \mathrm{d}\phi^- + \frac{\mathrm{i}g}{q\,[2]}\left(\sqrt{\frac{[4]}{[2]\,[3]}}\left[\frac{1}{2}\right]\left[\frac{3}{2}\right]B + qW^3\right)\phi^- + \frac{\mathrm{i}g\sqrt{2}}{q\,[2]}W^-\phi^0, \\ \mathrm{D}\phi^0 &= \mathrm{d}\phi^0 + \frac{\mathrm{i}g}{q\,[2]}\left(\sqrt{\frac{[4]}{[2]\,[3]}}\left[\frac{1}{2}\right]\left[\frac{3}{2}\right]B - \frac{1}{q}W^3\right)\phi^0 + \frac{\mathrm{i}g\sqrt{2}}{q\,[2]}W^+\phi^-. \end{aligned} \tag{4.21}$$

Suppose there exists a map $V : \Omega(\mathcal{M}) \to k$ such that $V(\Phi^\dagger\Phi)$ is hermitean, invariant, and bounded from below. If such a $V$ exists, it plays the role of a potential, and

$$S_{\mathrm{H}} = \langle (\mathrm{D}\Phi)^\dagger | \mathrm{D}\Phi \rangle - V(\Phi^\dagger\Phi) \tag{4.22}$$

is an invariant hermitean action for $\Phi$. Even though this action is manifestly $SU_q(2)$-invariant, assume that the quantum symmetry is broken spontaneously via a Higgs mechanism. This is accomplished by assuming that there is a real nonzero constant $v$ such that $V$ is minimized (and vanishes) at $\frac{1}{2}v^2 1$ (where 1 is the unit in $\Omega(\mathcal{M})$, which will be suppressed from now on). Therefore, the vacuum for the action above occurs at $\langle \Phi^\dagger\Phi \rangle = \frac{1}{2}v^2$, which one can assume corresponds to $\langle \phi^\pm \rangle = 0$ and $\langle \phi^0 \rangle = \langle \bar{\phi}^0 \rangle = (1/\sqrt{2})v$. (An example of a map satisfying all these conditions might be something like

$$V(\Phi^\dagger\Phi) := \frac{\mu^2}{v^2}\left\langle \Phi^\dagger\Phi - \frac{1}{2}v^2 \,\middle|\, \Phi^\dagger\Phi - \frac{1}{2}v^2 \right\rangle, \tag{4.23}$$

with $\mu \in \mathbb{R}$.)

If all the above are possible, then the action will acquire terms quadratic in the gauge fields, and thus they will become massive. Now, just as in the classical case, assume that the mass eigenstates are $W^\pm$, $Z$ and $A$, where

$$W^3 \equiv \cos\theta_W Z + \sin\theta_W A, \quad B \equiv -\sin\theta_W Z + \cos\theta_W A, \tag{4.24}$$

where $\theta_W \in \mathbb{R}$ is the Weinberg angle. However, in order for $A$ to be interpretable as the photon (more on this in the Section 4.4), it must be massless, which implies that $\mathrm{D}\Phi$ cannot include a term of the form $A\phi^0$ in $\mathrm{D}\phi^0$, since $\langle\phi^0\rangle \neq 0$. By using the explicit form (4.21) and the definitions of $Z$ and $A$ above, this may be accomplished by requiring

$$\tan\theta_W = q\sqrt{\frac{[4]}{[2]\,[3]}\begin{bmatrix}\frac{1}{2}\end{bmatrix}\begin{bmatrix}\frac{3}{2}\end{bmatrix}}. \tag{4.25}$$

Note two things: First of all, this expression for $\tan\theta_W$ is independent of the coupling constant $g$, unlike the classical case where it is given as the ratio of the $SU(2)$ and $U(1)$ couplings $g$ and $g'$. Secondly, even though $\tan\theta_W$ is still a function of $q$, if one assumes that the "real world" lives at (or at least very close to) $q = 1$, s/he therefore concludes that $\tan\theta_W = \sqrt{3/8}$, or alternatively, $\sin^2\theta_W = \frac{3}{11} = 0.273$. The experimental value is 0.2319 [24], so the predicted value is within 20%. (Recall that this is a tree-level prediction only; no mention has been made of quantum effects.)

By inserting the value for $\theta_W$ from the above relation, the covariant derivatives take the form

$$\begin{aligned}\mathrm{D}\phi^- &= \mathrm{d}\phi^- + \frac{\mathrm{i}g}{\cos\theta_W}\left(\frac{1}{[2]} - \sin^2\theta_W\right)Z\phi^- + \frac{\mathrm{i}g\sqrt{2}}{q\,[2]}W^-\phi^0 + \mathrm{i}g\sin\theta_W A\phi^-,\\ \mathrm{D}\phi^0 &= \mathrm{d}\phi^0 - \frac{\mathrm{i}g}{q^2\,[2]\cos\theta_W}Z\phi^0 + \frac{\mathrm{i}g\sqrt{2}}{q\,[2]}W^+\phi^-.\end{aligned} \tag{4.26}$$

An immediate result of this is that $\phi^-$ has electric charge $-g\sin\theta_W$. Assuming that this is equal to the charge of the electron, then by using the value of the fine structure constant $\alpha^{-1} = 137.04$ [24], one finds $g = \mathrm{e}/\sin\theta_W = \sqrt{44\pi\alpha/3} = 0.580$.

The masses of the three remaining gauge fields are found by evaluating $S_H$ at $\langle\Phi\rangle$, giving

$$S_H|_{\langle\Phi\rangle} = \frac{g^2v^2}{q^2\,[2]^2}\langle W^+|W^-\rangle + \frac{g^2v^2}{2q^4\,[2]^2\cos^2\theta_W}\langle Z|Z\rangle. \tag{4.27}$$

Requiring this to be equal to $m_W^2\langle W^+|W^-\rangle + \frac{1}{2}m_Z^2\langle Z|Z\rangle$ determines the masses:

$$m_W = \frac{gv}{q\,[2]} = qm_Z\cos\theta_W, \tag{4.28}$$

which gives $m_W \to m_Z\cos\theta_W$ in the classical case, so if $m_Z = 91.187$ GeV [24], $m_W = m_Z\cos\theta_W = \sqrt{8/11}m_Z = 77.76$ GeV. The experimental value is $m_W = 80.22$ GeV, which is about 3% away. Furthermore, the Higgs VEV is given by $v = (q^2\,[2]\sin 2\theta_W)/(2\sqrt{4\pi\alpha})m_Z$, which, after taking $q \to 1$ and sticking all the numbers in, is $v = 268.21$ GeV.

### 4.4. $U(1)$ Symmetry and electric charges

To address the question of whether or not there remains any symmetry after $\Phi$ gets a nonzero VEV, one can introduce two new fields $H$ and $\phi$ defined by

$$H := \sqrt{2}\left[\frac{1}{2}\right]\left(\bar{\phi}^0 + \frac{1}{q}\phi^0\right) - v, \qquad \phi := \frac{\sqrt{2}}{\mathrm{i}q}\left[\frac{1}{2}\right](\phi^0 - \bar{\phi}^0), \tag{4.29}$$

both of which have vanishing VEVs. Reexpressing (4.20) in terms of these new fields gives

$$\begin{aligned} H\phi^{\pm} &= \phi^{\pm}H + \mathrm{i}(1-q)\phi^{\pm}\phi, \\ \phi\phi^{\pm} &= \left(q + \frac{1}{q} - 1\right)\phi^{\pm}\phi + \mathrm{i}\left(1 - \frac{1}{q}\right)\phi^{\pm}H + \mathrm{i}\left(1 - \frac{1}{q}\right)v\phi^{\pm}, \\ H\phi &= \phi H + 2\mathrm{i}\left(1 - \frac{1}{q}\right)\phi^{+}\phi^{-}. \end{aligned} \tag{4.30}$$

Notice that the last term in the second of these is linear in $\phi^{\pm}$, whereas every other term is quadratic in the fields; this is what breaks the $SU_q(2)$ symmetry. In fact, the only coactions which preserve these commutation relations and are linear in the fields are

$$H \mapsto 1 \otimes H,\, \phi \mapsto 1 \otimes \phi,\, \phi^{\pm} \mapsto a^{\pm 1} \otimes \phi^{\pm}, \tag{4.31}$$

where $a$ is the sole generator of a QG with the relations $\Delta(a) = a \otimes a$, $\epsilon(a) = 1$, and $S(a) = \theta(a) = a^{-1}$. Note that this QG is entirely abelian, and thus is equivalent to the *classical* group $U(1)$.

One may define a new derivative $\mathrm{D}'$ by subtracting off the VEV of $\phi^0$ from (4.26), i.e.

$$\begin{aligned} \mathrm{D}'\phi^{-} &:= \mathrm{D}\phi^{-} - \frac{\mathrm{i}gv}{q\,[2]}W^{-}, \\ \mathrm{D}'(\phi^0 - \frac{1}{\sqrt{2}}v) &:= \mathrm{D}\phi^0 + \frac{\mathrm{i}gv}{q^2\sqrt{2}\,[2]\cos\theta_{\mathrm{W}}}Z. \end{aligned} \tag{4.32}$$

Under the remaining $U(1)$ symmetry, $\mathrm{D}'$ is in fact a covariant derivative, provided the gauge fields transform according to

$$W^{\pm} \mapsto \mathrm{e}^{\pm \mathrm{i}g\sin\theta_{\mathrm{W}}\chi} \otimes W^{\pm}, \quad Z \mapsto 1 \otimes Z, \quad A \mapsto 1 \otimes A + \delta\chi \otimes 1, \tag{4.33}$$

where $a := \mathrm{e}^{\mathrm{i}g\sin\theta_{\mathrm{W}}\chi}$. These are precisely the gauge transformations for a classical gauged $U(1)$, so the interpretation of $A$ as the photon is indeed justified after all.

At the QLA, rather than the QG, level, the nonvanishing of $\langle\phi^0\rangle$ means that the vacuum state $\langle\Phi\rangle$ is no longer null, and that one should look for all $2 \times 2$ matrices annihilating this state in order to see what remains after the spontaneous symmetry breaking. This is straightforward; all matrices proportional to $\mathrm{diag}(-1, 0)$. Now, note that if a quantum group $\mathcal{A}$ left coacts on a field $\psi$, then the dual QLA $\mathcal{U}$ is linearly right represented on this same field via

$$\psi \triangleleft x := \langle x, \psi^{(1)'}\rangle\psi^{(2)} \tag{4.34}$$

for any element $x \in \mathcal{U}$ (so that $\phi \triangleleft (xy) = (\phi \triangleleft x) \triangleleft y$). Therefore, in the case being considered, an element of $U_q(\mathfrak{su}(2))$ acts on $\Phi$ as

$$\Phi^i \triangleleft x = \langle x, U^i{}_j \rangle \Phi^j = \underline{\mathrm{fn}}^i{}_j(x) \Phi^j. \tag{4.35}$$

If $Q$ is the element of $U_q(\mathfrak{su}(2))$ which is equal to diag$(-1, 0)$ in the fundamental, then $Q \equiv (q/\lambda\,[2]\,[1/2]\,[3/2])T_0 + T_3$. This is central, so the remaining symmetry subalgebra is abelian, and is thus the classical algebra $\mathfrak{u}(1)$.

This can then be gauged, with the action of $Q$ on a field $\psi$ living in rep $\rho$ given by $\psi \triangleleft Q = \rho(Q)\psi$, and when $T_0$ is eliminated in favor of $T_3$ and $Q$, then the covariant derivative $\mathrm{D}'$ in rep $\rho$ takes the form

$$\begin{aligned} \mathrm{D}'\psi = \mathrm{d}\psi &- \frac{\mathrm{i}g\sqrt{2}}{[2]}(W^+\rho(T_+) + W^-\rho(T_-))\psi \\ &- \frac{\mathrm{i}g}{\cos\theta_{\mathrm{W}}} Z(\rho(T_3) - \sin^2\theta_{\mathrm{W}}\rho(Q))\psi - \mathrm{i}g\sin\theta_{\mathrm{W}} A\rho(Q)\psi. \end{aligned} \tag{4.36}$$

### *4.5. Fermions*

The undeformed SM includes fermionic matter, of course, so now an attempt is made to put the same into the deformed model considered herein. To this end, let $\Psi^i$ be a fermion doublet living in the fundamental, with $\bar{\Psi}_i$ its adjoint; in components,

$$\Psi := \begin{pmatrix} \psi \\ \nu \end{pmatrix}, \quad \bar{\Psi} := (\,\bar{\psi} \;\; \bar{\nu}\,). \tag{4.37}$$

Assume that there is a covariant derivative $\not{\mathrm{D}}$ for fermions which preserves the transformation properties (which in the familiar undeformed case is the usual $\not{\mathrm{D}} := \gamma^\mu \mathrm{D}'_\mu$, but here it is just assumed to exist without specifying its particular form). Then, if the contribution to the total action is taken to be $S_{\mathrm{F}} := \langle \bar{\Psi} | i\not{\mathrm{D}}\Psi \rangle$, one finds

$$\begin{aligned} S_{\mathrm{F}} = \langle \bar{\psi} | \mathrm{i}\not{\partial}\psi \rangle + \langle \bar{\nu} | \mathrm{i}\not{\partial}\nu \rangle - g\sin\theta_{\mathrm{W}} \langle \bar{\psi} | \not{A}\psi \rangle - \frac{g\sqrt{2}}{q\,[2]}(\langle \bar{\psi} | \not{W}^-\nu \rangle + \langle \bar{\nu} | \not{W}^+\psi \rangle) \\ + \frac{g}{\cos\theta_{\mathrm{W}}}\left(\left(-\frac{1}{[2]} + \sin^2\theta_{\mathrm{W}}\right)\langle \bar{\psi} | \not{Z}\psi \rangle + \frac{1}{q^2\,[2]}\langle \bar{\nu} | \not{Z}\nu \rangle\right). \end{aligned} \tag{4.38}$$

So, $\psi$ has electric charge $-g\sin\theta_{\mathrm{W}}$ and $\nu$ is neutral.

In principle, the $W - \nu - \psi$ coupling will result in a four-fermion interaction in the low-energy theory, as in the classical case, and thus would give the Fermi coupling constant $G_{\mathrm{F}}$. In the $q \to 1$ limit, by using the value of $g$ from before, this gives a prediction of $G_{\mathrm{F}} := g^2/4\sqrt{2}m_W^2 = 121\pi\alpha/24\sqrt{2}m_Z^2 = 0.983 \times 10^{-5}\ \mathrm{GeV}^{-2}$, about 16% away from the value in [24], $1.16639 \times 10^{-5}\ \mathrm{GeV}^{-2}$.

## 5. Conclusions

This model, despite some interesting features, is still very minimal; it cannot be considered a truly deformed version of the SM as it stands. There are still several areas which must be looked into if there is any hope whatsoever of treating the results presented above seriously. Some of these concerns will now be briefly addressed.

### 5.1. *Mathematical concerns*

The actual geometrical interpretation of non(anti)commuting differential forms as introduced here is a bit unclear. Suppose one does in fact have the four connection 1-forms satisfying relations (4.1); classically, it is natural to immediately write $\Gamma^A := \Gamma^A_\mu \mathrm{d}x^\mu$, where the $\{x^\mu\}$ are a set of local coordinates on the space–time considered. However, how is the noncommutative nature of the connections manifested? Do the individual components $\{\Gamma^A_\mu\}$ commute, but the coordinates do not, as in the case of the so-called Manin plane [25]? Or is the space–time classical and the components form some nontrivial algebra [26]? Or both? Or neither? Or can one even hope to interpret the connections in this way?

Then there is the question of what this mysterious quadratic form $\langle \mid \rangle$ actually is; in the classical case, as already mentioned, it is just an integration over the space–time. To extend this to the QG case, one needs to understand how to integrate over noncommuting objects. In the case of strictly *anticommuting* variables, this was done long ago with the Berezin integral [27], and there has also been extensive work done for the general noncommuting situation (cf. [28] and references therein). Thus, it seems like the possibility of building actions using these latter types of integrals may exist, as it did in the case of supersymmetry using the former.

### 5.2. *Physical concerns*

#### 5.2.1. *Chiral matter*

One obvious physical shortcoming of this model as a truly deformed version of the SM is the fact that it seems to preclude the inclusion of chiral fermions. After all, this is one of the reasons the SM has a $SU(2) \times U(1)$ symmetry, as opposed to something like a $U(2)$ symmetry: The left-handed fermions live in the $(\frac{1}{2}, 0)$ rep and the right-handed ones in the $(0, Y)$ rep ($Y$ being the $U(1)$ hypercharge of the fermion). In the model herein, it looks like the only possible way to incorporate chirality would be to have the left-handed fermions in a $SU_q(2)$ doublet and the right-handed ones in a singlet, i.e. the trivial rep. Unfortunately, it would seem that this is problematic, because the trivial rep is given by the vanishing of all the generators, and would therefore give no coupling of the right-handed fermions to the gauge fields.

However, all may not be lost; it is true that *a* 1-dimensional rep of $SU_q(2)$ has all the generators vanishing. But recall (3.6) and (3.7): These relations are also satisfied by the instance where $T_\pm$ and $T_3$ vanish, but $T_0$ is represented by $2\,[2]/\lambda$. Thus, if $\chi$ is a fermion living in this "trivial" rep $\underline{\mathrm{tv}}$, its contribution to the action may be taken to be

$$\begin{aligned}\langle\bar{\chi}|\mathrm{i}\underline{\mathrm{tv}}(\not{D})\chi\rangle &= \langle\bar{\chi}|\mathrm{i}\not{\partial}\chi\rangle + \frac{2\mathrm{i}\,[2]}{\lambda}\langle\bar{\chi}|\not{\Gamma}^0\chi\rangle \\ &= \langle\bar{\chi}|\mathrm{i}\not{\partial}\chi\rangle - \frac{g}{\cos\theta_\mathrm{W}}\left(\frac{2\sin^2\theta_\mathrm{W}}{q\lambda^2\,[1/2]\,[3/2]}\right)\langle\bar{\chi}|\not{Z}\chi\rangle \\ &\quad + g\sin\theta_\mathrm{W}\left(\frac{2}{q\lambda^2\,[1/2]\,[3/2]}\right)\langle\bar{\chi}|\not{A}\chi\rangle. \end{aligned} \tag{5.1}$$

So this assumption does indeed allow one to couple the fermion with the $Z$ and the photon *without* coupling it to $W^{\pm}$, exactly what one would want for a right-handed fermion.

However, there is still the matter of combining such a fermion with a doublet; the obvious thing to try would be to take the singlet and doublet fermions given by

$$\chi := \tfrac{1}{2}(1+\gamma_5)\psi, \quad \Psi := \tfrac{1}{2}(1-\gamma_5)\begin{pmatrix}\psi \\ \nu\end{pmatrix}. \tag{5.2}$$

Then, one would want to manipulate the kinetic energy terms for each of these (given by (4.38) and (5.1)) and collect all the terms quadratic in $\psi$ together in such a way that the $Z$ and photon couplings work out (the $W^{\pm}$ coupling between $\psi$ and $\nu$ already have the correct form). The author has so far been unsuccessful in this; perhaps some readers of this paper may be able to accomplish this task.

### 5.2.2. *Quarks*

Even if the chiral problem can be solved, one must still try to incorporate all the observed matter into the model, which must include quarks. As in the classical case, these must have an $SU(3)$ color symmetry (which would remain undeformed, presumably), but also have to have appropriate $SU_q(2)$ properties. There may already be some hint as to what these properties might be: Remember (3.8), where there is an explicit relation between the u(1) charge $Q$, the "$z$-spin component" $T_3$ and the casimir $J^2$. In particular, in the $q \to 1$ limit, $J^2 = \frac{3}{2}(T_3 - Q)$ so that a state in the spin-$j$ irrep of $SU(2)$ with $z$-component quantum number $m$ has

$$Q_{j,m} = m - \tfrac{2}{3}j(j+1). \tag{5.3}$$

(This replaces the classical Gell-Mann–Nishijima relation $Q = T_3 + \frac{1}{2}Y$.) For $j = \frac{1}{2}$, this gives charges of 0 for the $m = +\frac{1}{2}$ state and $-1$ for the $m = -\frac{1}{2}$ state, as desired. But what about other reps? The next one to consider is the adjoint, i.e. $j = 1$, for which $Q_{1,m} = m - \frac{4}{3}$. For $m = (+1, 0, -1)$, the charges are $(-\frac{1}{3}, -\frac{4}{3}, -\frac{7}{3})$. Even though no known multiplet has such charges, the appearance of the 3 in the denominators is intriguing, and indicates that perhaps there is some way of putting quarks into the theory via the adjoint rep.

### 5.2.3. *Higgs potential and couplings*

The Higgs potential $V(\Phi^{\dagger}\Phi)$ was just included with the assumption that it has a minimum which will give a nonzero VEV to $\Phi$, without anything being said about its actual form (although an example was given). This unfortunately looks like it would have to be put in by hand in this model, unlike in [10], where it arises automatically as a consequence of the noncommutative nature of the model.

How about couplings between the Higgs and fermions? One might initially try to find invariant combinations of the Higgs doublet and a fermion doublet $\Psi$ as in (4.37). One springs immediately to mind: The obvious one given by $\bar{\Psi}\Phi \equiv \bar{\psi}\phi^- + \bar{\nu}\phi^0$. Another can be constructed following the classical example, using the deformed Levi-Civita symbol for $SU_q(2)$, denoted by $\epsilon_q^{ij}$. This object has the values

$$\epsilon_q^{12} = q^{\frac{1}{2}}, \quad \epsilon_q^{21} = -q^{-\frac{1}{2}}, \quad \epsilon_q^{11} = \epsilon_q^{22} = 0 \tag{5.4}$$

so that the unit determinant condition on the $2 \times 2$ $SU_q(2)$ matrices can then be written as

$$\epsilon_q^{k\ell} U^i{}_k U^j{}_\ell \equiv \epsilon_q^{ij} 1. \tag{5.5}$$

This means that one can define the doublet $\tilde{\Phi}^i$ by

$$\tilde{\Phi}^i := \epsilon_q^{ji} \Phi_j^\dagger = \begin{pmatrix} -q^{\frac{1}{2}} & \bar{\phi}^0 \\ q^{-\frac{1}{2}} & \phi^+ \end{pmatrix}. \tag{5.6}$$

Under the action of the QG, this transforms as ${}_A\Delta(\tilde{\Phi}^i) = U^i{}_j \otimes \tilde{\Phi}^j$, so the combination $\bar{\Psi}\tilde{\Phi}$ is also invariant under the QG.

Classically, one would then take one of these two combinations and construct a charge-conserving interaction term by using a fermion of opposite chirality. For example, if $\chi$ is the right-handed partner of $\psi$, then

$$\langle \bar{\Psi}\tilde{\Phi}|\chi\rangle = q^{-\frac{1}{2}}\langle \bar{\nu}\phi^+|\chi\rangle - q^{\frac{1}{2}}\langle \bar{\psi}\bar{\phi}^0|\chi\rangle \tag{5.7}$$

would seem to be a possible Yukawa coupling between the chiral fermions and the Higgs. Unfortunately, if $\chi$ is thought of as a fermion in the "trivial" rep suggested above, then this term is *not* $SU_q(2)$ invariant, since $T_0$ is nonzero in this rep. One actually wants a combination of $\bar{\Psi}$ and $\Phi$ which also lives in this "trivial" rep, and transforms in such a way so as to cancel out the variation of $\chi$. The author has yet to find such a combination, so the coupling of the Higgs to any chiral fermions is still absent in this model.

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## Appendix A. $SU_q(2)$ Connection commutation relations

### A.1. Connection 1-Forms

$$\begin{aligned}
&\Gamma^\pm \wedge \Gamma^\pm = 0,\\
&\Gamma^\pm \wedge \Gamma^3 + q^{\pm 2}\Gamma^3 \wedge \Gamma^\pm = 0,\\
&\Gamma^\pm \wedge \Gamma^0 + \Gamma^0 \wedge \Gamma^\pm = \pm\frac{q^{\pm 1}\lambda}{[2]}\Gamma^3 \wedge \Gamma^\pm,\\
&\Gamma^+ \wedge \Gamma^- + \Gamma^- \wedge \Gamma^+ = -q^2\,[2]^2\,\Gamma^0 \wedge \Gamma^0,\\
&\Gamma^0 \wedge \Gamma^3 + \Gamma^3 \wedge \Gamma^0 = -\frac{\lambda}{q}\Gamma^- \wedge \Gamma^+ - \frac{\lambda\,[2]}{q}\Gamma^0 \wedge \Gamma^0,\\
&\Gamma^3 \wedge \Gamma^3 = \frac{\lambda\,[2]}{q}\Gamma^- \wedge \Gamma^+ - \frac{[2]^2}{q^2}\Gamma^0 \wedge \Gamma^0.
\end{aligned}$$

### A.2. $d\Gamma$ Commutation Relations

$$\begin{aligned}
&\mathrm{d}\Gamma^0 \wedge \Gamma^a = \Gamma^a \wedge \mathrm{d}\Gamma^0,\\
&\mathrm{d}\Gamma^\pm \wedge \Gamma^\pm - \Gamma^\pm \wedge \mathrm{d}\Gamma^\pm = 0,\\
&\mathrm{d}\Gamma^\pm \wedge \Gamma^\mp - \Gamma^\mp \wedge \mathrm{d}\Gamma^\pm\\
&\quad = \pm q\lambda\Gamma^0 \wedge \mathrm{d}\Gamma^3 \pm \frac{q\lambda}{[2]}\Gamma^3 \wedge \mathrm{d}\Gamma^3 \mp \lambda\,[2]\,\Gamma^0 \wedge \Gamma^- \wedge \Gamma^+,\\
&\mathrm{d}\Gamma^\pm \wedge \Gamma^3 - \Gamma^3 \wedge \mathrm{d}\Gamma^\pm\\
&\quad = \mp q^{\pm 1}\lambda\Gamma^\pm \wedge \mathrm{d}\Gamma^3 \mp q^{\pm 1}\lambda\,[2]\,\Gamma^0 \wedge \mathrm{d}\Gamma^\pm - q^{\pm 2}\lambda\,[2]\,\Gamma^0 \wedge \Gamma^3 \wedge \Gamma^\pm,\\
&\mathrm{d}\Gamma^\pm \wedge \Gamma^0 - (1+\lambda^2)\Gamma^0 \wedge \mathrm{d}\Gamma^\pm\\
&\quad = \mp\frac{q^{\mp 1}\lambda}{[2]}\Gamma^3 \wedge \mathrm{d}\Gamma^\pm \pm \frac{q^{\pm 1}\lambda}{[2]}\Gamma^\pm \wedge \mathrm{d}\Gamma^3 \pm q^{\pm 1}\lambda^2\Gamma^0 \wedge \Gamma^3 \wedge \Gamma^\pm,\\
&\mathrm{d}\Gamma^3 \wedge \Gamma^\pm - \Gamma^\pm \wedge \mathrm{d}\Gamma^3\\
&\quad = \pm q^{\mp 1}\lambda\Gamma^3 \wedge \mathrm{d}\Gamma^\pm \pm q^{\mp 1}\lambda\,[2]\,\Gamma^0 \wedge \mathrm{d}\Gamma^\pm + \lambda\,[2]\,\Gamma^0 \wedge \Gamma^3 \wedge \Gamma^\pm,\\
&\mathrm{d}\Gamma^3 \wedge \Gamma^3 - (1-\lambda^2)\Gamma^3 \wedge \mathrm{d}\Gamma^3\\
&\quad = \frac{\lambda\,[2]}{q}\Gamma^+ \wedge \mathrm{d}\Gamma^- - \frac{\lambda\,[2]}{q}\Gamma^- \wedge \mathrm{d}\Gamma^+\\
&\qquad - \lambda^2\,[2]\,\Gamma^0 \wedge \mathrm{d}\Gamma^3 + \frac{\lambda^2\,[2]^2}{q}\Gamma^0 \wedge \Gamma^- \wedge \Gamma^+,\\
&\mathrm{d}\Gamma^3 \wedge \Gamma^0 - (1+\lambda^2)\Gamma^0 \wedge \mathrm{d}\Gamma^3\\
&\quad = \frac{\lambda}{q}\Gamma^- \wedge \mathrm{d}\Gamma^+ - \frac{\lambda}{q}\Gamma^+ \wedge \mathrm{d}\Gamma^- + \frac{\lambda^2}{[2]}\Gamma^3 \wedge \mathrm{d}\Gamma^3 - \frac{\lambda^2\,[2]}{q}\Gamma^0 \wedge \Gamma^- \wedge \Gamma^+,\\
&\mathrm{d}\Gamma^3 \wedge \mathrm{d}\Gamma^\pm - q^{\pm 2}\mathrm{d}\Gamma^\pm \wedge \mathrm{d}\Gamma^3\\
&\quad = \pm q^{\pm 1}\lambda\,[2]\,\mathrm{d}\Gamma^0 \wedge \mathrm{d}\Gamma^\pm + q^{\pm 2}\lambda\,[2]\,\Gamma^3 \wedge \Gamma^\pm \wedge \mathrm{d}\Gamma^0
\end{aligned}$$

$$
\begin{aligned}
&\quad - q^{\pm 2}\lambda\,[2]\,\Gamma^0 \wedge \Gamma^{\pm} \wedge \mathrm{d}\Gamma^3 + \lambda\,[2]\,\Gamma^0 \wedge \Gamma^3 \wedge \mathrm{d}\Gamma^{\pm},\\
&\mathrm{d}\Gamma^+ \wedge \mathrm{d}\Gamma^- - \mathrm{d}\Gamma^- \wedge \mathrm{d}\Gamma^+\\
&\quad = q\lambda \mathrm{d}\Gamma^0 \wedge \mathrm{d}\Gamma^3 + \frac{q\lambda}{[2]}\mathrm{d}\Gamma^3 \wedge \mathrm{d}\Gamma^3 + \lambda\,[2]\,\Gamma^0 \wedge \Gamma^+ \wedge \mathrm{d}\Gamma^-\\
&\quad - \lambda\,[2]\,\Gamma^0 \wedge \Gamma^- \wedge \mathrm{d}\Gamma^+ - \lambda\,[2]\,\Gamma^- \wedge \Gamma^+ \wedge \mathrm{d}\Gamma^0\\
&\quad - q\lambda^2 \Gamma^0 \wedge \Gamma^3 \wedge \mathrm{d}\Gamma^3.
\end{aligned}
$$

### A.3. Field strength commutation relations

$$
\begin{aligned}
&F^0 \wedge F^a = F^a \wedge F^0,\\
&F^3 \wedge F^{\pm} - q^{\pm 2} F^{\pm} \wedge F^3 = \pm q^{\pm 1}\lambda\,[2]\,F^0 \wedge F^{\pm},\\
&F^+ \wedge F^- - F^- \wedge F^+ = q\lambda F^0 \wedge F^3 + \frac{q\lambda}{[2]} F^3 \wedge F^3.
\end{aligned}
$$

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